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Domain Wall Fermions and the η-Invariant

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We extend work by Callan and Harvey and show how the phase of the chiral fermion determinant in four dimensions is reproduced by zero modes bound to a domain wall in five dimensions. The analysis could shed light on the applicability of zero mode fermions and the vacuum overlap formulation of Narayanan and Neuberger for chiral gauge theories on the lattice.

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1. Introduction

In ref. [1] Callan and Harvey analyzed fermion zero modes in background gauge fields bound to domain wall and vortex defects in arbitrary numbers of dimensions. The case of the domain wall is particularly interesting, since an effective chiral theory in even dimensions can be embedded in an odd dimensional Dirac theory. The purpose of ref. [1] was to elucidate how anomalies of the effective theory arose due to Chern-Simons currents in the full theory. It was subsequently shown that the same system could be implemented on the lattice without encountering doubling of the chiral mode spectrum [2]; Chern-Simons current flow occurs on the lattice as well [3,4] giving rise to an anomalous current divergence in the effective theory along the defect. The discussion motivated the vacuum overlap formulation of chiral gauge theories on the lattice of Narayanan and Neuberger [5] who proposed a specific ansatz for the chiral fermion determinant both on the lattice and in the continuum as the overlap of two particular quantum states. The vacuum overlap formalism was subsequently used to compute the nonabelian anomaly in four dimensions, using a lattice regulator [6] as well as the Lorentz anomaly in two-dimensional quantum gravity [7].

Most discussions of the domain wall system have focussed on the anomaly as a test of the chiral nature of the effective theory (for a recent exception, see [8]). In this Letter we return to the continuum to extend the Callan Harvey results beyond the anomaly to include the complete phase of the chiral fermion determinant in gauge backgrounds of trivial topology. In particular we show that in taking care to properly regulate the theory, one can reproduce the η-invariant formulation of the chiral phase as developed by Alvarez-Gaumé *et al.* [9] (see also [10]). Although we will restrict our attention to the continuum, it is our hope that the analysis can eventually shed light on lattice chiral gauge theory.

2. A continuum path integral for chiral fermions

In four Euclidian dimensions the determinant of a Dirac fermion coupled to background gauge fields $\det(i\not{D})$ can be defined as a positive definite object (by using Pauli-Villars regulators, for example). The determinant of a chiral fermion $\det(i\not{D}P_L)$ is formally

given by

$$\det(i\not{D}P_L) \equiv Z[A] \equiv e^{-W[A]} = e^{-i\phi[A]}\sqrt{\det(i\not{D})} \tag{2.1}$$

Thus it is the phase $\operatorname{Im} W[A]$ which we wish to understand; it contains all of the anomalies in the theory as well as other physics. In this section we will try to establish the connection between the phase $\operatorname{Im} W[A]$ and the phase of the determinant for a five-dimensional Dirac fermion in the presence of a domain wall.

Consider the Euclidian path integral $\mathcal{Z}[\mathcal{A}]$ corresponding to a fermion in five dimensions with a space dependent mass in a background gauge field $\mathcal{A}$ that has the particular form of a four-dimensional gauge field embedded in five dimensions:

$$\mathcal{A}_\mu = (A_i(x), 0) \ . \tag{2.2}$$

If the mass m only depends on the fifth coordinate s, then

$$\mathcal{Z}[\mathcal{A}] = \mathrm{e}^{-\mathcal{W}[\mathcal{A}]} = \int [\mathrm{d}\Psi][\mathrm{d}\overline{\Psi}] \exp\left(-\mathcal{S}[\mathcal{A}]\right) \tag{2.3}$$

where

$$\mathcal{S}[\mathcal{A}] = \int \mathrm{d}^5x\, \overline{\Psi} K[\mathcal{A}]\Psi \ , \qquad K[\mathcal{A}] = i\gamma_5\partial_s - im(s) + i\not{D}_4 \ . \tag{2.4}$$

In the above expression $\not{D}_4$ is the four dimensional covariant derivative, independent of s. Four- and five-dimensional tensor indices are denoted by Roman and Greek letters respectively. The letters $\mathcal{A}$, $\mathcal{S}$, $\mathcal{Z}$ and $\mathcal{W}$ refer to five dimensional gauge field, action, partition function and effective action respectively, while A, S, Z, W denote their four-dimensional analogues. We take $\mathcal{A}$ to be Hermitian and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Capital $\Psi(x,s)$ denotes a four component Dirac spinor in five dimensions, while lowercase $\psi(x)$ will be a four component, four dimensional Dirac spinor.

Since the operator $K[\mathcal{A}]$ is separable, it is convenient to expand the Ψ fields in a product basis of the form

$$\begin{aligned}
\Psi(x,s) &= \sum_n P_L\psi_n(x) b_n(s) + \sum_n P_R\psi_n(x) f_n(s) \ , \\
\overline{\Psi}(x,s) &= \sum_n \overline{\psi}_n(x) P_R b_n^*(s) + \sum_n \overline{\psi}_n(x) P_L f_n^*(s) \ .
\end{aligned} \tag{2.5}$$

where the $\psi_n(x)$ are arbitrary four dimensional Dirac spinors, and $P_{R,L} = (1 \pm \gamma_5)/2$. The functions $b_n(s)$ and $f_n(s)$ are taken to satisfy the eigenvalue equations

$$\begin{aligned} \left[-\partial_s^2 + m(s)^2 + \dot{m}(s)\right] f_n(s) &= \mu_n^2 f_n(s) \ , \\ \left[-\partial_s^2 + m(s)^2 - \dot{m}(s)\right] b_n(s) &= \mu_n^2 b_n(s) \ . \end{aligned} \tag{2.6}$$

In the above equation, $\dot{m} \equiv \partial_s m$ and the spectrum μ_n is assumed to be discrete. For nonzero μ_n, the b_n and f_n are paired; however, there can be an arbitrary number of unrelated b and f zero modes as well.

The equations (2.6) can be regarded as the Schrödinger equation for a supersymmetric quantum mechanical system, with supersymmetry generator $Q = [\partial_s + m(s)]\gamma_0 P_L$. The functions $b_n P_L$ and $f_n P_R$ correspond to the "boson" and "fermion" eigenstates of $\{Q, Q^\dagger\}$ which are necessarily degenerate when they have nonzero "energy" μ_n^2. The bosonic and fermionic "vacuum states" (zero modes) need not be related however [11].

The five dimensional action (2.4) can be recast in this basis as a four dimensional action involving an infinite number of flavors:

$$S[A] = \int \mathrm{d}^4x \left[\sum_{k=1}^{n_b} \overline{\psi}_k i\not{D}_4 P_L \psi_k + \sum_{k=1}^{n_f} \overline{\psi}_k i\not{D}_4 P_R \psi_k + \sum_n \overline{\psi}_n (i\not{D}_4 - i\mu_n)\psi_n \right] \ . \tag{2.7}$$

where n_b and n_f are the number of bosonic and fermionic zero mode solutions to eq. (2.6) respectively, and the final sum excludes the zero modes. Provided this action can be suitably regulated, we see that it corresponds to an infinite tower of massive Dirac fermions with mass μ_n, as well as n_b left handed and n_f right handed chiral fermions. If we chose $m(s)$ and the boundary conditions to eq. (2.6) such that the supersymmetry is unbroken, we are guaranteed that there will be at least one chiral fermion, corresponding to the groundstate of the supersymmetric Hamiltonian.

The system Callan and Harvey considered consisted of a step function for the mass $m(s)$ in infinite volume. There was a single zero mode in the action (2.7), and they showed that the anomaly in the zero mode current was compensated by the Chern-Simons current induced by integrating out the heavy fermions in (2.7). Instead, we wish to use the five dimensional system to define the full chiral phase $\mathrm{Im}\, W[A]$. It is impossible to do so with the system considered in [1] which lacks both regulators and boundary conditions

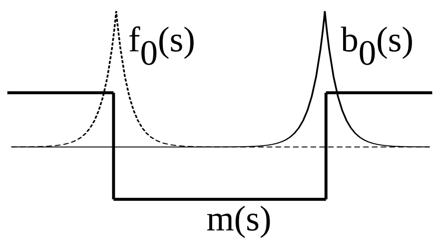

Fig. 1. A domain wall/anti-wall pair at $s = \pm L_s/2$ respectively with periodic boundary condtions at $s = \pm L_s$ for the functions $b(s)$ and $f(s)$ in eq. (2.6). The lowest eigenstates are a pair of exact zero modes, with the boson localized at $s = L_s/2$ (solid line) and the fermion localized at $s = -L_s/2$ (dashed line).

in the fifth dimension; we are forced to introduce both [1]. We will work in finite volume and use conventional Pauli-Villars regulators, which requires an action that has an equal number of right and left handed fields. In particular, we choose the mass function $m(s)$ to represent a domain wall – anti-wall pair with periodic boundary conditions on the b and f eigenfunctions; the result is a theory with a discrete spectrum and a pair of exact zero mode solutions to eq. (2.6), f_0 and b_0 pictured in fig. 1.

Wave function renormalization diagrams for gauge bosons in five dimensions are linearly divergent, but since there is no subleading logarithmic divergence only one Pauli-Villars field is required to regulate the effective action $\mathcal{W}[\mathcal{A}] - \mathcal{W}[0]$. We take the regulator to have mass $(m(s) + M)$ and loop factor $+1$, where a normal fermion has loop factor -1. The regulated fermion determinant takes the form

$$(\det K)_{reg.} = \frac{\det K}{\det(K - iM)} \ . \tag{2.8}$$

We will be examining the phase of this object in the limit that the regulator mass M is taken to be large.

Until now we have only considered gauge fields $\mathcal{A}$ independent of the coordinate s. However, with s-independent gauge fields, both zero modes f_0 and b_0 couple equally,

[1] Ref. [1] ignored the regulators since the induced Chern-Simons current is finite; however one finds that the regulators needed to make sense out of the rest of the theory make finite contributions to the current. See [3] for example.

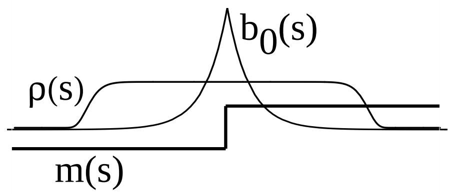

Fig. 2. Profile in the fifth dimension of the mass function $m(s)$ and the gauge field weighting function $\rho(s)$, which equals one for $|s - L_s/2| < L_s/2$ and goes to zero outside that region. $\rho(s)$ is chosen so that it is nonzero in the support of the chiral mode $b_0(s)$, but vanishes in the vicinity of the mode $f_0(s)$.

and the theory we are describing is vector-like and incapable of reproducing the four-dimensional chiral phase. This is the price of introducing conventional Pauli-Villars regulators. However one can take advantage of the fact that the two zero modes f_0 and b_0 are spatially separated in the fifth dimension, and only couple one of them to the gauge field. This is accomplished by modifying eq. (2.2), coupling the fermions to a particular five-dimensional gauge field $\hat{\mathcal{A}}_\mu$ of the form

$$\hat{\mathcal{A}}_i = A_i(x)\rho(s) \ , \qquad \hat{\mathcal{A}}_5 = 0 \ , \tag{2.9}$$

where $\rho(s)$ is a real function of s with $\rho(s) = 1$ in the region $|s - L_s/2| < L_s/2$, smoothly going to zero outside that region. The region where $\rho = 1$ is chosen to include the massless "boson" state at $s = L_s/2$ as pictured in fig. 2, but not to overlap with the massless "fermion" state at $s = -L_s/2$.

Since $\hat{\mathcal{A}}$ is s-dependent, the fermion operator $K[\hat{\mathcal{A}}]$ is no longer separable, and we must deal directly with the regulated five dimensional effective action $\mathcal{W}[\hat{\mathcal{A}}]$ rather than treating the theory as four dimensional theory with an infinite tower of flavors. The central assertion of this paper is that the phase $\mathrm{Im}\, W[A]$ of the chiral determinant in four dimensions with gauge field $A_i(x)$ is given by

$$\mathrm{Im}\, W[A] - \mathrm{Im}\, W[0] = \mathrm{Im}\, \mathcal{W}[\hat{\mathcal{A}}] - \mathrm{Im}\, \mathcal{W}[0] \tag{2.10}$$

where $\hat{\mathcal{A}}$ is the particular five dimensional gauge field given in eq. (2.9). We first show that eq. (2.10) correctly reproduces the anomaly *à la* Callan-Harvey [1], and then we prove

that in fact it reproduces the complete phase as expressed in papers by Alvarez-Gaumé *et al.*

3. Gauge invariance and the anomaly

In this section we give a heuristic argument along the lines of ref. [1] for why our expression for the four dimensional chiral determinant (2.10) correctly reproduces the anomalous phase under four dimensional gauge transformations. A more rigorous argument follows in the next section.

Define the infinitesimal four- and five-dimensional gauge transformations δ_4 and δ_5:

$$\delta_4 A_i(x) = -D_i\theta(x) = -(\partial_i\theta + [A_i, \theta]) \ , \tag{3.1}$$

$$\delta_5 \mathcal{A}_\mu(x,s) = -D_\mu v(x,s) = -(\partial_\mu v + [\mathcal{A}_\mu, v]) \ . \tag{3.2}$$

Gauge invariance of the five-dimensional theory implies

$$\delta_5 \,\mathrm{Im}\, \mathcal{W}[\mathcal{A}] = 0 \ . \tag{3.3}$$

We wish to prove that when we vary our ansatz $W[A]$ under a four-dimensional gauge transformation, we get the correct consistent anomaly for a four-dimensional chiral fermion [10]:

$$\delta_4 \,\mathrm{Im}\, W[A] \equiv \delta_4 \,\mathrm{Im}\, \mathcal{W}[\hat{\mathcal{A}}] = \frac{1}{24\pi^2}\int \mathrm{d}^4x \mathrm{Tr}\left[\theta \mathrm{d}(A\mathrm{d}A + \tfrac{1}{2}A^3)\right] \ . \tag{3.4}$$

The reason why five-dimensional gauge invariance in eq. (3.3) does not imply four-dimensional gauge invariance of $W[A]$ is that with $\hat{\mathcal{A}} = (A_i(x)\rho(s), 0)$, $\delta_4\hat{\mathcal{A}} \equiv (\delta_4 A_i \rho, 0)$ cannot be written as $\delta_5\hat{\mathcal{A}}$.

Inspired by ref. [1] we write the effective action for an arbitrary five-dimensional gauge field $\mathcal{A}$ as the sum of contributions from the chiral mode plus the contributions from everything else — the heavy modes:

$$\mathcal{W}[\mathcal{A}] = \mathcal{W}_\chi[\mathcal{A}] + \mathcal{W}_h[\mathcal{A}] \ .$$

From fig. 2 we see that the chiral mode is localized near the domain wall at $s = L/2$ where $\rho = 1$ and $\dot{\rho} = 0$. Over that region the four dimensional gauge transformation δ_4 is identical to a five dimensional gauge transformation δ_5 with $v(x,s) = \theta(x)\rho(s)$, therefore

$$\delta_4 \,\mathrm{Im}\, \mathcal{W}_\chi[\hat{\mathcal{A}}] = \delta_5 \,\mathrm{Im}\, \mathcal{W}_\chi[\hat{\mathcal{A}}]\Big|_{v(x,s)=\theta(x)\rho(s)} = -\delta_5 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}]\Big|_{v(x,s)=\theta(x)\rho(s)} \tag{3.5}$$

where the last equality follows from five dimensional gauge invariance (3.3). This allows us to express $\delta_4 \,\mathrm{Im}\, \mathcal{W}[\hat{\mathcal{A}}]$ in terms of $\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}]$ alone

$$\begin{aligned}\delta_4 \,\mathrm{Im}\, W[A] = \delta_4 \,\mathrm{Im}\, \mathcal{W}[\hat{\mathcal{A}}] &= \delta_4 \,\mathrm{Im}\, \mathcal{W}_\chi[\hat{\mathcal{A}}] + \delta_4 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}] \\ &= -\delta_5 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}]\Big|_{v=\theta\rho} + \delta_4 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}] \ .\end{aligned} \tag{3.6}$$

We now determine the two terms separately, using an explicit expression for $\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}]$ which we calculate perturbatively in the adiabatic limit, following [1]. In the limit of large domain wall mass and smooth gauge fields the lowest dimensional operator in the adiabatic expansion is [2]

$$\mathrm{Im}\, \mathcal{W}_h[\mathcal{A}] = -\pi \int \mathrm{d}^5 x \left(\frac{m(s)}{|m(s)|} - \frac{M}{|M|} \right) Q_5^0[\mathcal{A}] \tag{3.7}$$

where Q_5^0 is the Cern-Simons form

$$Q_5^0[\mathcal{A}] = \frac{1}{(2\pi)^3 2!} \int_0^1 \mathrm{d}\sigma \ \mathrm{Tr}\left[\mathcal{A}(\sigma \mathrm{d}\mathcal{A} + \sigma^2 \mathcal{A}^2)^2\right] \ . \tag{3.8}$$

It is easily verified by substitution that

$$\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}] = 0 \qquad \text{and} \qquad \delta_4 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}] = 0$$

for the gauge field configuration $\hat{\mathcal{A}}$ defined in (2.9). Calculating the δ_5 term in (3.6) yields

$$\begin{aligned}&-\delta_5 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}]\Big|_{v=\theta\rho} \\ &= \frac{1}{24\pi^2} \int \mathrm{d}^4 x \, \mathrm{d}s \left(\delta(s - \frac{L_s}{2}) - \delta(s + \frac{L_s}{2}) \right) \mathrm{Tr}\left[v \mathrm{d}(\hat{\mathcal{A}} \mathrm{d}\hat{\mathcal{A}} + \tfrac{1}{2}\hat{\mathcal{A}}^3) \right] \Big|_{v=\theta\rho} \\ &= \frac{1}{24\pi^2} \int \mathrm{d}^4 x \mathrm{Tr}\left[\theta \mathrm{d}(A \mathrm{d}A + \tfrac{1}{2} A^3) \right] \ .\end{aligned} \tag{3.9}$$

In the final step above we used the fact that $\rho(L_s/2) = 1$, $\rho(-L_s/2) = 0$. Substituting this result and $\delta_4 \,\mathrm{Im}\, \mathcal{W}_h[\hat{\mathcal{A}}] = 0$ into (3.6) we finally obtain the result we set out to prove in eq. (3.4) — namely, that the variation of our ansatz (2.10) for $W[A]$ under four-dimensional gauge transformations correctly reproduces the consistent non-Abelian anomaly for a chiral fermion.

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[2] See also refs. [12-13] for more detailed computations than found in ref. [1].

4. The complete chiral phase and the η invariant

We now arrive at the central point of this paper, which is to show that $\operatorname{Im} W[A]$ as defined in eq. (2.10) reproduces not just the anomaly, but the entire phase of the chiral determinant (in the zero instanton sector). In refs. [9,10] it was shown that the chiral phase can be related to certain properties of the five-dimensional operator

$$\mathcal{H} = i\partial_s\gamma_5 + i\not{D}[\mathcal{A}_i] \tag{4.1}$$

where $\mathcal{A}_5 = 0$ and $\mathcal{A}_i(x,s)$ describes a path in the space of four-dimensional gauge fields from $\mathcal{A}_i(x,-\infty) = \overline{A}_i(x)$ to $\mathcal{A}_i(x,+\infty) = A_i(x)$. The four-dimensional gauge field $\overline{A}(x)$ plays the role of some fiducial gauge field which is left fixed, while A can be varied. In particular, ref. [9] showed that in the zero instanton sector, the effective action for a chiral fermion in four dimensions may be expressed as

$$\operatorname{Im}(W[A] - W[\overline{A}]) = \pi(\eta[\mathcal{H}] + \text{dim Ker } \mathcal{H}) - 2\pi Q_5^0[\mathcal{A}_i]\ . \tag{4.2}$$

Q_5^0 is the Chern-Simons form given in eq. (3.8); $\eta[\mathcal{H}]$ is the so-called η-invariant of $\mathcal{H}$, defined to be the regulated sum of the signs of its eigenvalues, $\sum \lambda/|\lambda|$, where possible zero modes are omitted in this definition. The contribution of $\mathcal{H}$ zero modes are accounted for by the (dim Ker $\mathcal{H}$) term. The combination of η and Q_5^0 is independent of the path $\mathcal{A}$, and only depends on the endpoints $\overline{A}$ and A [14]. The authors of ref. [9] further showed that the η-invariant can be expressed as

$$\begin{aligned}\eta[\mathcal{H}] &= -\frac{1}{\pi}\lim_{M\to\infty} \operatorname{Im}\operatorname{Tr}\ln\left[\frac{\mathcal{H}-iM}{\mathcal{H}+iM}\right] \\ &= -\frac{1}{\pi}\lim_{M\to\infty}\lim_{L\to\infty}\int_0^1 \mathrm{du}\ \frac{\mathrm{d}}{\mathrm{du}}\operatorname{Im}\operatorname{Tr} P_L \ln\left[\frac{\mathcal{H}^u-iM}{\mathcal{H}^u+iM}\right]\ ,\end{aligned} \tag{4.3}$$

where $P_L(s) = \theta(s+L)\theta(L-s)$, and zero modes of $\mathcal{H}$ are omitted. P_L is inserted to keep track of the noncompactness of the manifold by providing explicit boundaries which are removed to infinity. The object $\mathcal{H}^u$ has been introduced so that the endpoint of the path in gauge field space can be smoothly changed as a function of the parameter u: $\mathcal{H}^u = \mathcal{H}[\mathcal{A}_i^u]$ with $\mathcal{A}_i^u(x,-\infty) = \overline{A}_i(x)$ and $\mathcal{A}_i^u(x,+\infty) = A_i^u(x)$ with $A_i^{u=0}(x) = \overline{A}_i(x)$ and $A_i^{u=1}(x) = A_i(x)$.

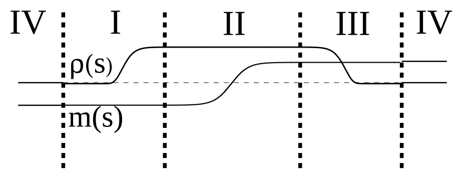

Fig. 3. The fifth dimension is divided into four regions for the purpose of computing $\mathrm{Im}\, W[A]$ *in eq.(4.5). The divisions occur far from where either* $\rho(s)$ *or* $m(s)$ *are varying. The gauge field (4.4) is independent of* u *in region IV and therefore that region does not contribute.*

We now demonstrate that the path integral for domain wall fermions (eq. (2.10)) can be rewritten in the form of the right side of eq. (4.3), thereby reproducing the result [9] of Alvarez-Gaumé *et al.* To achieve this, we use the Pauli-Villars regulated expression (2.8) for the five-dimensional determinant and consider the particular family of paths in field space of the form

$$\begin{aligned} \hat{\mathcal{A}}^u_\mu(x,s) &= (A^u_i(x)\rho(s) + \overline{A}_i(x)(1-\rho(s)), 0) \ , \\ A_i^{u=0} = \overline{A}_i(x) \ , &\qquad A_i^{u=1}(x) = A_i(x) \end{aligned} \tag{4.4}$$

where $\rho(s)$ is the smooth function discussed previously and pictured in fig. 2. For the wall/anti-wall configuration pictured in fig. 1 we can now write $\mathrm{Im}\,\mathcal{W}$ as

$$\begin{aligned} \mathrm{Im}(\mathcal{W}[\hat{\mathcal{A}}^{u=1}] - \mathcal{W}[\hat{\mathcal{A}}^{u=0}]) &= \mathrm{Im} \int_0^1 \mathrm{du}\, \frac{\mathrm{d}}{\mathrm{du}} \left(\ln\det \left[\frac{K[\hat{\mathcal{A}}^u]}{K[\hat{\mathcal{A}}^u] - iM} \right] \right) \\ &= \int_0^1 \mathrm{du}\, \frac{\mathrm{d}}{\mathrm{du}}\, \mathrm{Im}\,\mathrm{Tr} \left[(P_I + P_{II} + P_{III}) \ln \frac{K[\hat{\mathcal{A}}^u]}{K[\hat{\mathcal{A}}^u] - iM} \right] \end{aligned} \tag{4.5}$$

in the appropriate limits that the fermion and Pauli-Villars masses and the box size L_s are taken to infinity. To compute the integral in (4.5) we have divided the fifth dimension into four regions, as shown in fig. 3. This is done by inserting $1 = P_I + P_{II} + P_{III} + P_{IV}$ into the trace defining the effective action W, where $P_R = 1$ in region R and $P_R = 0$ elsewhere. Region IV includes the "fermionic" mode bound to the anti-domain wall (not pictured), corresponding to a right-handed chiral fermion, and $\rho(s) = 0$ in this region. Thus $\frac{\mathrm{d}}{\mathrm{du}}\hat{\mathcal{A}}^u_\mu = 0$ and so region IV does not contribute to the expression (4.5).

We now compute the contributions to the trace from each of the regions I–III. Note that our operator $K[\mathcal{A}^u]$ may be written in terms of the Alvarez–Gaumé *et al.* Hamiltonian

(4.1) as

$$K[\mathcal{A}^u] = \mathcal{H}^u - im(s)\ .$$

In regions I and III the step function mass $m(s)$ equals $\mp m_0$ so that $K[\mathcal{A}_i^u] = \mathcal{H}^u \pm im_0$ respectively, and note that the paths (4.4) – restricted to regions I and III – are examples of the paths used in the expression for the η invariant (4.3). In region I we have

$$\begin{aligned} &\mathrm{Im}\,\mathrm{Tr}\left(P_I \ln \frac{K}{K - iM}\right) \\ &\quad = -\frac{i}{2}\,\mathrm{Tr}\,P_I \left(\ln \frac{\hat{\mathcal{H}}^u + im_0}{\hat{\mathcal{H}}^u - im_0} - \ln \frac{\hat{\mathcal{H}}^u + i(m_0 - M)}{\hat{\mathcal{H}}^u - i(m_0 - M)}\right)\ . \end{aligned} \tag{4.6}$$

Comparing the above expression with the expression (4.3) for the η-invariant, we find

$$\begin{aligned} &\lim_{m_0\to\infty}\lim_{M\to\infty}\lim_{L_s\to\infty}\int_0^1 \mathrm{du}\,\frac{\mathrm{d}}{\mathrm{du}}\,\mathrm{Im}\,\mathrm{Tr}\left(P_I \ln \frac{K}{K - iM}\right) \\ &\quad = (1+1)\frac{\pi}{2}(\eta[\mathcal{H}] + \dim\,\mathrm{Ker}\,\mathcal{H}) = \pi(\eta[\mathcal{H}] + \dim\,\mathrm{Ker}\,\mathcal{H})\ , \end{aligned} \tag{4.7}$$

where we have made explicit the equal contributions from the fermion and Pauli-Villars fields. Note the ordering of limits: first we must send the wall/antiwall separation to infinity; then we take the regulator mass to infinity; and finally we take the domain wall height m_0 to infinity. This ensures that first the interaction between the gauge field and the unwanted zero mode at $s = -L_s/2$ is sent to zero; then the Pauli-Villars fields are decoupled from the five dimensional theory; and finally the heavy five-dimensional fermion modes are decoupled from the effective four dimensional theory on the domain wall.

The signs of the fermion and Pauli-Villars contributions to the above expression are determined from (4.6) by the signs of the masses; thus in region III where the step function is positive, the fermion contribution changes sign and the two fields give a combined contribution $\propto (-1+1) = 0$ — there is no contribution from this region to $\mathrm{Im}\,W[A]$ [3].

The contribution from region II — where the fermion mass is changing, but the gauge field is constant — vanishes as well. To see this we write the contribution as

$$\mathrm{Im}\,\mathrm{Tr}\left(P_{II} \ln \frac{K}{K - iM}\right) = \mathrm{Im}\int_0^M \mathrm{d}z\,\mathrm{Tr}\left(P_{II}\frac{i}{K - iz}\right)\ . \tag{4.8}$$

[3] This behaviour is directly related to the fact that when the theory is regulated, the Chern-Simons current flow only comes from region I (with twice the value found in ref. [1]) — see eq. (3.7).

But the operator $\mathcal{O} \equiv i\not{D}\gamma_5$ satisfies $\mathcal{O}K\mathcal{O}^{-1} = -K^\dagger$ and $[\mathcal{O}, P_{II}] = 0$; thus

$$\begin{aligned}
\mathrm{Im}\,\mathrm{Tr}\left(P_{II}\frac{i}{K-iz}\right) &= \mathrm{Im}\,\mathrm{Tr}\left(\mathcal{O}\mathcal{O}^{-1}P_{II}\frac{i}{K-iz}\right) \\
&= \mathrm{Im}\,\mathrm{Tr}\,P_{II}\left(\frac{i}{K-iz}\right)^\dagger \\
&= 0\ .
\end{aligned} \tag{4.9}$$

It follows that the five-dimensional path integral for domain wall fermions in the sequential limits of eq. (4.7) has the phase

$$\mathrm{Im}(\mathcal{W}[\hat{\mathcal{A}}^{u=1}] - \mathcal{W}[\hat{\mathcal{A}}^{u=0}]) = \pi(\eta[\mathcal{H}] + \dim\,\mathrm{Ker}\ \mathcal{H}) \tag{4.10}$$

where $\mathcal{H}$ is the operator (4.1) as a functional of the gauge path $\hat{\mathcal{A}}_\mu = (A_i\rho + \overline{A}_i(1-\rho), 0)$ acting in region I in fig. 3, in the limit that the region is infinite in extent. The right hand side of (4.10) agrees with the general result of Alvarez-Gaumé *et al.* (4.2), provided that the Cern-Simon form $Q_5^0[\hat{\mathcal{A}}]$ vanishes. If we restrict ourselves to paths $\hat{A}$ with $\overline{A} = 0$, then $Q_5^0[\hat{\mathcal{A}}] = 0$ and we have proven our central assertion (2.10) that five dimensional domain wall fermions with $\overline{A} = 0$ correctly reproduce the complete chiral phase in the zero instanton sector. (Note that $\overline{A} = 0$ implies $\hat{\mathcal{A}}^{u=0} = 0$).

It would be very interesting to pursue this analysis beyond the zero instanton sector, but we do not do so here.

Finally, we mention that the domain wall formula correctly reproduces the SU(2) Witten anomaly [11]. Consider an $SU(2)$ gauge theory with an odd number of $SU(2)$-doublet Weyl fermions. For real and pseudo-real representations $Q_5^0[\hat{A}] = 0$, and so the domain wall result (4.10) reproduces the exact result (4.2) for any gauge field $\overline{A}$. If the gauge field $\overline{A} = A^g$ is related to A by a large gauge transformation g which is a nontrivial element of $\pi_4(SU(2))$, then by the Atiyah-Singer theorem, $(\mathrm{dimKer}\mathcal{H})$ is an odd integer. Furthermore, $\eta(\mathcal{H})$ vanishes because for real (and for pseudo-real) representations the nonzero eigenvalues of $\mathcal{H}$ come in opposite sign pairs [9]. Thus our formula (4.10) correctly reproduces Witten's anomaly [11]

$$\mathrm{Im}\,W[A] = \mathrm{Im}\,W[A^g] + \pi\ ,$$

and the fermion determinant picks up a minus sign under the large nontrivial gauge transformation.

5. Relation to the vacuum overlap formulation

Motivated by domain wall fermions, Narayanan and Neuberger have suggested that one can write $\det(i\not{D}P_L)$ as the overlap between ground states of different Hamiltonians:

$$\frac{\det(i\not{D}P_L)}{\det(i\not{\partial}P_L)} = \frac{\langle A-|\,A+\rangle}{\langle 0-|\,0+\rangle} \tag{5.1}$$

where the states $|A\pm\rangle$ refer to the ground states of the two different four-dimensional Hamiltonians $H^{\pm}$ in the same background gauge field $A_i(x)$:

$$H^{\pm}[A] = i\not{D}_4[A] \mp im_0$$

in the limit $m_0 \to \infty$, with the phase convention that $\langle 0+|\,A+\rangle$ and $\langle 0-|\,A-\rangle$ are real [5]. The correct anomalous transformation of this phase has been computed in the continuum (without regulators) in 1+1 dimensions for an Abelian theory by Narayanan and Neuberger [15], and for a non-Abelian theory in $3+1$ dimensions by Randjbar-Daemi and Strathdee [6].

What we will now show is that for the zero instanton gauge backgrounds that we are considering in this Letter, the vacuum overlap expression (5.1) reproduces not just the anomaly, but the complete phase of the chiral determinant. For such gauge fields, $|A\pm\rangle$ can be written as [5]

$$\lim_{S\to\infty} N e^{-SH^{\pm}}\,|0\pm\rangle \tag{5.2}$$

where the normalization constant N depends on both S and A, but is real. Thus the chiral phase predicted by the overlap formulation can be written in path integral form as

$$\operatorname{Im} W[A] - \operatorname{Im} W[0] = \int [\mathrm{d}\Psi][\mathrm{d}\overline{\Psi}]\, e^{-\int_{-\infty}^{\infty}\mathrm{d}s\,\int \mathrm{d}^4x\,\mathcal{L}[A]} \tag{5.3}$$

where $\mathcal{L}[A]$ is the Lagrange density

$$\mathcal{L}[A] = \overline{\Psi}(x,s)\left[i\gamma_5\partial_s + i\gamma_j(\partial_j + iA_j(x)\overline{\rho}(s)) - i\overline{m}(s)\right]\Psi(x,s) \tag{5.4}$$

with a step function mass $\overline{m}(s) = m_0\epsilon(s)$, and a gauge function $\overline{\rho}(s)$ which is of the form

$$\overline{\rho}(s) = \lim_{s_0\to\infty} \theta(s_0+s)\theta(s_0-s)\ . \tag{5.5}$$

Evidently the phase of the vacuum overlap formula in the zero instanton sector is a special case of our formula for domain wall fermions derived in the previous section, involving a particular choice for the functions $m(s)$ and $\rho(s)$. It follows that the ansatz (5.1) also correctly reproduces the full chiral phase. The appeal of the overlap formulation is that it can be implemented on the lattice in a straightforward fashion, and that it can be used to calculate Green functions in gauge backgrounds of nontrivial topology.

6. Conclusions

We have shown how to relate the phase of the chiral fermion determinant in a background non-Abelian gauge field to the phase of a five-dimension path integral over Dirac fermions interacting with a domain wall. In particular, we have demonstrated how the η-invariant description of the chiral phase due to Alvarez-Gaumé *et al.* arises when the theory is properly regulated with well defined boundary conditions on the fields. The phase we have calculated includes not only the perturbative anomaly, but also possible Witten anomalies [16], as well as non-anomalous contributions. The result was seen to extend to the vacuum overlap formulation of chiral fermions.

A serious deficiency in our arguments is that they only apply to the topologically trivial gauge sector; it would be interesting if the arguments could be extended to discuss the 't Hooft vertex in the presence of instantons. While the domain wall model allows one to compute the chiral phase, it is less evident how to use it to define fermion Green functions for a chiral theory.

Application of these ideas to the calculation of the chiral phase in a lattice regulated theory is promising since the periodic boundary conditions we employed are simple to implement and fermion doublers can be eliminated by means of a Wilson term [2]. The tricky part of a lattice realization of domain wall fermions lies in the limiting process in eq. (4.7). A crucial step in the analysis was that the regulator mass M was taken to infinity with fixed background gauge field, so that gauge fields and their derivatives were always small compared to M. On a lattice, the role of M is played by the inverse lattice spacing for the fermion fields. However, in the usual formulation of lattice gauge theories, the

spatial variation of the gauge field is set by the same lattice spacing, and so the limiting process needed for reproducing the correct phase is not obtainable [4]. However, the problem would seem to be resolved if the gauge fields were integrated over a coarser lattice than the fermions. The limit $a_\psi/a_U \to 0$ for the ratio of the lattice spacings is analogous to the $M \to \infty$ limit in the Pauli-Villars regulated continuum theory. This suggestion is in line with a recently proposed solution to the chiral fermion problem by Hernandez and Sundrum [17]. We believe that this would also alleviate problems with the domain wall formulation discussed recently in ref. [18].

Some of the delicate limiting procedures necessary for using domain wall fermions on the lattice might be evaded by pursuing the vacuum overlap formulation of chiral gauge theories, rather than its domain wall progenitor. It would be useful to better understand the connection between the two approaches, particularly for topologically nontrivial gauge fields.

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[4] One might think that in the weak coupling limit, only smooth gauge fields would be present; however, gauge symmetry is explicitly broken in the domain wall formulation, as discussed in §3, and the fermions are sensitive to the wildly fluctuating gauge fields found on every gauge orbit.

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